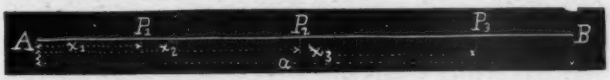
With reference to the positions of the points relative to each other six mutually exclusive cases are equally possible, namely,

$$x_1 < x_2 < x_3,$$
$$x_1 < x_3 < x_2,$$
$$x_2 < x_3 < x_1,$$
$$x_2 < x_1 < x_3,$$
$$x_3 < x_1 < x_2,$$
$$x_3 < x_2 < x_1,$$

consequently the total number of favorable cases is equal to six times the number of favorable cases—subject to one of these conditions, say to $x_1 < x_2 < x_3$.

Reasoning precisely as in the case of two points we see that the possibility that $P_1$, $P_2$, $P_3$ are at the distances $x_1$, $x_2$, $x_3$ from $A$ is $dx_1 dx_2 dx_3/a^3$, and that the required probability is the triple integral of this expression, the limits of integration being so chosen as to include all favorable and to exclude all unfavorable positions. For $x_1 < x_2 < x_3$ these limits are

for $P_3$ from $x_2 + b$ to $a$,
for $P_2$ from $x_1 + b$ to $a - b$,
for $P_1$ from 0 to $a - 2b$.




$$\therefore P = \frac{6}{a^3}\int_0^{a-2b}\int_{x_1+b}^{a-b}\int_{x_2+b}^{a} dx_1 dx_2 dx_3 = \int_0^{a-2b}\int_{x_1+b}^{a-b}(a-b-x_2)\,dx_1 dx_2$$

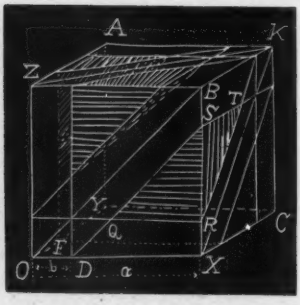
$$= \frac{3}{a^3}\int_0^{a-2b}(a-2b-x_1)^2\,dx_1 = \left(\frac{a-2b}{a}\right)^3.$$

Second Solution. Consider a cube of side $a$. Take one set of concurrent edges of this cube for axes of coördinates and denote by $x$, $y$, $z$ the coördinates of any point $P$ within this cube. To every triplet of points $P_1$, $P_2$, $P_3$ on the line $AB$ corresponds a single point in the cube, viz, the point whose coördinates are $x = x_1$, $y = x_2$, $z = x_3$; to every point in the cube corresponds a single triplet of points on the line, viz, the points whose distances from $A$ are respectively $x_1 = x$, $x_2 = y$, $x_3 = z$. The one-to-one correspondence between the points in the cube and the different portions of three points on the line being established, the problem is solved by comparing the volume of the regions of the cube for which the coördinates of the points satisfy the simultaneous conditions

the difference between $x$ and $y > b$,
the difference between $y$ and $z > b$,
the difference between $z$ and $x > b$,

with the volume of the entire cube.

Take $OD=OE=OF=b$, and through each of the points $F$ and $D$ draw planes parallel to the diagonal plane $OZKC$.




For every point between the planes thus drawn we have the difference between $x$ and $y<b$; for every point in these planes the difference between $x$ and $y=b$; and for the points exterior to the planes we have the difference between $x$ and $y>b$. Again, if planes be drawn through each of the points $E$ and $F$ parallel to the diagonal plane $OXKA$, we see that the difference between $y$ and $z>b$ only for points which lie exterior to both of these planes, and finally in order that the difference between $z$ and $x>b$ the points must be exterior to the planes drawn through $E$ and $D$ respectively, parallel to $OYKB$. The points which satisfy simultaneously the three conditions

the difference between $x$ and $y>b$,
the difference between $y$ and $z>b$,
the difference between $z$ and $x>b$,

are therefore limited to those regions of the cube which lie exterior to each of these three sets of parallel planes, that is to six equal tetrahedrons such as $QRST$.

The dimensions of this tetrahedron are $QR=RS=ST=a-2b$, since $BS=OD=a$ and also $RX=OE=a$, and the volume of $QRST=\frac{1}{6}(a-2b)^3$.

The required probability is therefore

$$P=\frac{6QRST}{\text{volume of entire cube}}=\left(\frac{a-2b}{a}\right)^3.$$

We will now prove the general

Theorem. *The probability that of n points, distributed at random along a line of length a, no two shall fall within the distance b of each other is*

$$\left[\frac{a-(n-1)b}{a}\right]^n.$$

We denote by $x_1, x_2, x_3, \ldots\ldots, x_n$ the distances of the $n$ points from one extremity of the line $a$, and first compute the probability that the distance between every pair of points exceeds $b$ when the points are subject to the condition

$$x_1<x_2<x_3<\ldots\ldots<x_n.$$

Denote this probability by $Q$. Then since the suffixes 1, 2, 3, ....., $n$, may be permuted in $n!$ number of ways, each permutation giving rise to a different distribution of points equally likely, we shall have for the required probability $P=n!Q$. Reasoning exactly as in the case of three points we see that

$$Q=\int_0^{a-(n-1)b}\int_{x_1+b}^{a-(n-2)b}\cdots\cdots\int_{x_{n-2}+b}^{a+b}\int_{x_{n-1}+b}^{a}\frac{dx_1}{a}\frac{dx_2}{a}\cdots\cdots\frac{dx_{n-1}}{a}\frac{dx_n}{a},$$

or if we drop suffixes, since there is no danger of confusion,

$$Q=\int_0^{a-(n-1)b}\int_{x+b}^{a-(n-1)b}\cdots\cdots\int_{x+b}^{a-b}\int_{x+b}^{a}\frac{dx^n}{a^r}.$$

To evaluate this integral we observe that in the evaluation of

$$\int_{x+b}^{a-2b}\int_{x+b}^{a-b}\int_{x+b}^{a}dx^3=\int_{x+b}^{a-2b}\int_{x+b}^{a-b}(a-b-x)dx^2=\int_{x+b}^{a-2b}\tfrac{1}{2}(a-2b-x)^2dx$$

$$=\frac{1}{3!}(a-3b-x)^3,$$

occur the expressions $a-b-x$, $\frac{1}{2}(a-2b-x)^2$, $\frac{1}{3!}(a-3b-x)^3$.

Let $u_k=\frac{1}{k!}(a-kb-x)^k$, $u_0=1$. Then

$$\int_{x+b}^{a-kb}u_k dx=\int_{x+b}^{a-kb}\frac{1}{k!}(a-kb-x)^k dx=\frac{1}{(k+1)!}[a-(k+1)b-x]^{k+1}=u_{k+1},$$

and hence we have the chain of expressions,

$$u_{n-1}=\int_{x+b}^{a-(n-2)b}u_{n-2}dx,\quad u_{n-2}=\int_{x+b}^{a-(n-3)b}u_{n-3}dx,\ \ldots\ldots,\ u_1=\int_{x+b}^{a}u_0 dx=\int_{x+b}^{a}dx.$$

Eliminating successively the quantities $u_{n-2}$, $u_{n-3}$, ....., $u_1$ on the right, and substituting for $u_{n-1}$ its value, we obtain

$$u_{n-1}=\int_{x+b}^{a-(n-2)b}\cdots\cdots\int_{x+b}^{a-b}\int_{x+b}^{a}dx^n=\frac{1}{(n-1)!}[a-(n-1)b-x]^{n-1},$$

$$Q=\frac{1}{a^n}\int_0^{a-(n-1)b}u_{n-1}dx=\frac{1}{a^n}\int_0^{a-(n-1)b}[a-(n-1)b-x]^{n-1}dx$$

$$=\frac{1}{a^n}\cdot\frac{1}{n!}[a-(n-1)b]^n.$$

$$\therefore P=n!\,Q=\left(\frac{a-(n-1)b}{a}\right)^n.$$

Corollary I. The probability that of $n$ points, distributed at random on a line of length $a$, at least two shall fall within a distance $b$ of each other is $1-P$.

Corollary II. The probability $p$ that of $n$ points, distributed at random on a closed curve of length $a$, no two fall within an arc $b$ of each other is

$$p=\left(\frac{a-nb}{a}\right)^{n-1}.$$

For let one point be located at random and let the arc be cut at that point and straightened. No point which falls within a distance $b$ of either end of this line gives a favorable case. We may therefore consider the favorable cases in the distribution of the remaining $n-1$ points on the line of length $a-2b$. The probability therefore involves one less integral sign and the limit of variation is not $a$ but $a-2b$. The denominator however remains $a$, since this is the measure of the total number of possible cases. Replacing therefore $n$ by $n-1$ in the expression $P$, and the $a$ in the numerator by $a-2b$ we obtain $p$.

Corollary III. The probability that of $n$ points distributed at random on a closed curve of length $a$, at least two fall within a distance $b$ of each other is $1-p$.

## NUMERALS FOR SIMPLIFYING ADDITION.

By R. A. HARRIS, Washington, D. C.

Primitive numerals are composed of the sign for unity repeated as many times as there are units in the numbers represented. Systems in which all nine figures are primitive were used by the Egyptians in their hieroglyphic writing, the Phœnicians, the Babylonians in their cuneiform inscriptions, the early Romans, and probably the early Indians. By virtue of such a notation, the process of addition can be reduced to counting by ones. The meaning of the symbols being self-evident does not have to be learned.*

In many instances, the first two, three, or four numerals are essentially primitive while the others are more arbitrary or superficial; *e. g.*, the hieratic symbols of the Egyptians, the ordinary Roman numrals, the moderately early Indian numerals, and the numerals now used by the Chinese.

The Palmyrenes made use of two component signs in constructing the first nine numerals,—simple strokes representing the ones and a certain Y-like symbol the component five found in each of the numbers five, six, seven, eight, and nine. Here addition can be performed by counting by ones and by fives,—the fives, for convenience, being taken in pairs so that most of the counting is really by tens instead of by fives. The same is true of the Roman numerals of the republican period where IIII is written for IV and VIIII for IX.

*Columns 3, 5, 6, and 7 of the accompanying figure are copied from the Encyclopedia Britannica, article "Numerals."

The Syrians made use of three component signs; one for unity, another for two, and another for five. Here addition can be performed by counting by ones, twos, and fives (but for the most part, of course, by tens instead of fives).

Although our numerals underwent considerable change in form prior to the invention of the art of printing, there appears to be no doubt that the Europeans received them (zero included) from the Arabs, and that the Arabs received them from the Hindoos. The origin of these symbols is obscure. The most reasonable supposition seems to be that the significant digits (or at least those greater than three) were in the first place the initial letters of the Sanskrit words for the numbers which they represent.

Ingenious attempts have been made to show how each of these nine figures might have originally consisted of as many straight marks, conveniently joined together, as the number itself contains units; or how, instead of lines, angles were the quantities considered. History, however, does not confirm either of these hypotheses, although the first may apply to the figures one, two, and possibly three. But whatever their origin, our figures became, centuries ago, mere signs of numbers conveying no suggestion of decomposition into component parts.*

In the second column of the accompanying table is a proposed set of figures, in which the component ones are vertical strokes, the twos horizontal strokes, and the fives closed loops. These figures are fairly distinctive however turned, can be made with comparative ease, and possess the important property of being obviously composed of ones, twos, and fives, as were the Syrian characters.

The stroke constituting figure two has been slightly curved so as to better distinguish it from the minus sign or the dash, and to prevent its being mistaken for figure one when it happens to be turned into the vertical. The strokes in figure four are likewise curved to distinguish the latter from the sign of equality, and to prevent confusion between the symbols for four and eleven.

The figures one, six, and eight of the proposed system closely resemble their ordinary or Arabic forms.

A convenient rule for adding a column of moderate length may be given thus: Count by twos the horizontal marks (all terminating to the right); upon this sum count the ones, known by the vertical marks; upon this result count by fives the closed loops, generally by tens (the loops being taken in pairs. The result of these three counts is the sum of the column.

To add the next column (to the left), begin by counting by twos if the

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*The following references are to works in which the history of numerals is considered:

Hankel: *Zur Geschichte der Mathematik in Altertum und Mittelalter.*

Brooks: *The Philosophy of Arithmetic*, pp. 21-31, 101-107, 141-146.

*Encyclopedia Britannica*, under "Numerals" and "Inscriptions."

Cantor: *Vorlesungen ueber Geschichte der Mathematik*, Vol. I, and Plate at end of Vol. I.

Fine: *The Number-System of Algebra*, pp. 79-90.

Ball: *A Short Account of the History of Mathematics*, 2d Ed., pp. 123-131, 141, 158, 161, 189-191.

Fink: *A Brief History of Mathematics*, pp. 6-17.

*Encyclopedie des Sciences Mathematiques*, I, 1, 1, pp. 20-21.

number carried be even. If it be odd, before proceeding mark a component one of the column to be added by placing a dot above it and count one upon the odd number carried thus making it even. Then proceed as in the first column, but taking care to omit the dotted component one when the ones are counted.

If the person adding can count equally well the odd numbers by twos, or if there should happen to be no component one in the column to be added, the borrowing of a one will not be resorted to.

In the case of a very long column, it may be best to simply count the

| Ordinary | Proposed | Syriac | Roman Rep. Period | Palmyrene | Phoenician | Hieroglyphic |
|---|---|---|---|---|---|---|
| 0 | ↄ | | | | | |
| 1 | / | / | I | / | I | I |
| 2 | ~ | ٢ | II | // | II | II |
| 3 | L | ٣ | III | /// | III | III |
| 4 | ≈ | ٤ | IIII | //// | IIII | IIII |
| 5 | o | ٥ | V | ⊃ | IIIII | IIIII |
| 6 | b | ٦ | VI | /⊃ | IIIIII | IIIIII |
| 7 | σ | ٧ | VII | //⊃ | IIIIIII | IIIIIII |
| 8 | ∂ | ٨ | VIII | ///⊃ | IIIIIIII | IIIIIIII |
| 9 | Œ | ٩ | VIIII | ////⊃ | IIIIIIIII | IIIIIIIII |
| 10 | ↄ | ١ | X | ⊃ | ⊃ | ∩ |

ones, twos, and fives (not *by* twos and *by* fives), to then multiply the number of twos by two, the number of fives by five, and finally to add together the three partial sums. This last step is supposed to be done as side work of which only the total sum is to be preserved.

These numerals are proposed for the purpose of saving mental labor in performing addition. The time required in adding by the above methods is about the same as that required by the nsual method, or possibly a little greater, but the mental strain is much less because of the symmetry of the process. The liability to mistakes is somewhat less and for the same reason.

Many subtractions are self-evident; *e. g.*, 87−52=35. Here the compo-

nents of the subtrahend are simply stricken from the minuend, and the result of this cancellation is the difference required, *i. e.*, $87-52=35$.

If characters like those proposed were in use, it seems probable that children could learn addition tables with far greater ease and certainty than is the case at present.

Finally, it may be remarked that if addition be carried out in the ordinary manner, these characters will be quite as serviceable as are the Arabic numerals, and being adapted to the mode of addition described above, a new and convenient way of proving the work will be afforded.

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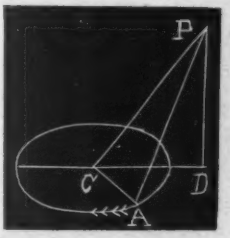
## THE CONVEX SURFACE OF AN OBLIQUE CONE.

By F. P. MATZ, Reading, Pa.

Legendre, in his *Théorie des fonctions elliptiques*, has devised a method for finding the convex surface of an oblique cone having a circular base.

We propose the following method: Let $CA=r$, $PD=a$, and $\angle PCD=\omega$. Represent the uniformly varying $\angle DCA$ by $\theta$; then since $CD=a\cot\omega$, we have by trigonometry,

$$\cos\theta=\frac{a^2\cot^2\omega+r^2-(AD)^2}{2ar\cot\omega}\ldots\ldots(1).$$




$$\therefore AD=\sqrt{(a^2\cot^2\omega+r^2-2ar\cot\omega\cos\theta)}\ldots\ldots(2),$$

and $$AP=\sqrt{(a^2\operatorname{cosec}^2\omega+r^2-2ar\cot\omega\cos\theta)}\ldots\ldots(3),$$

which *line* represents an element of convex surface of the oblique cone.

$$\therefore S=2r\int_0^{\pi}\sqrt{(a^2\operatorname{cosec}^2\omega+r^2-2ar\cot\omega\cos\theta)}\,d\theta\ldots\ldots(4).$$

Transforming (4) under the supposition that $\theta=\pi+2\phi$, we get

$$S=2r\int_0^{\frac{1}{2}\pi}\sqrt{(a^2\operatorname{cosec}^2\omega+r^2+2ar\cot\omega\cos2\phi)}\times2d\phi\ldots\ldots(5),$$

$$=4r\int_0^{\frac{1}{2}\pi}\sqrt{(a^2\operatorname{cosec}^2\omega+r^2+2ar\cot\omega-2ar\cot\omega\sin^2\phi)}\,d\phi\ldots\ldots(6),$$

$$=4r\int_0^{\frac{1}{2}\pi}\sqrt{\left[\left(\frac{a^2+2ar\sin\omega\cos\omega+r^2\sin^2\omega}{\sin^2\omega}\right)-\left(\frac{2ar\sin\omega\cos\omega}{\sin^2\omega}\right)\sin^2\phi\right]}\,d\phi\ldots\ldots(7).$$

Representing *in order* these parenthetical expressions by $m^2$ and $n^2$, (7) gives

$$S=4mr\int_0^{\frac{1}{2}\pi}\sqrt{[1-(n^2/m^2)\sin^2\phi]}d\phi \quad \ldots\ldots(8);$$

that is, according to Legendre's system of notation for elliptic functions,

$$n^2/m^2=k^2=\frac{2ar\sin\omega\cos\omega}{a^2+2ar\sin\omega\cos\omega+r^2\sin^2\omega} \quad \ldots\ldots(9).$$

By means of (9), we have from (8),

$$S=4mr\int_0^{\frac{1}{2}\pi}\sqrt{(1-k^2\sin^2\phi)}d\phi=4mrE(k,\tfrac{1}{2}\pi)$$

$$=2\pi mr\left[1-\sum_{p=1}^{p=\infty}\left(\frac{1.3.5\ldots\ldots(2p-1)}{2.4.6\ldots\ldots\ldots 2p}\right)^2\left(\frac{k^{2p}}{2p-1}\right)\right]\ldots\ldots(10),$$

which is the formula required.

Corollary. If $\omega=90°$, $k^2=0$; and under this supposition, we have from (8),

$$S=4r\sqrt{(r^2+a^2)}\int_0^{\frac{1}{2}\pi}d\phi=2\pi r\sqrt{(r^2+a^2)},$$

which is the formula for the convex surface of a right circular cone.

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# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

Problem 99. March, 1899; March, 1900; April, 1904.

Solution by F. H. SAFFORD, Ph. D., University of Pennsylvania.

In the "seven" problem the writer has used the following method to show that there is a single solution, notation apart. Let the natural order *1234567* be contained in every solotion. There are fifteen arrangements containing *124* which do not conflict with the natural order. The same is true for *125*, and for *126*, while there are seventeen for *127*. Advancing the figures by two units, the arrangements containing *346*, *348*, *341*, *342* are obtained; similarly for *561*, etc., then for *713*, etc., ending with the sets containing *672*, *673*, *674*, *675*. In this

way all but twenty-three arrangements out of the complete number of possible arrangements have been written down, and these are also tabulated. Returning to the arrangements in which *34* appears, all which contain *12* are struck out. From those containing *56*, all which contain *12* or *34* are struck out, similarly throughout the other arrangements. There are 225 sets of three arrangements each, in which besides the natural order there may be written an arrangement containing *124* and one containing *125*. But only about one hundred of these are admissible since the others fail owing to conflicting "triads" in the second and third arrangements. These sets of three arrangements are next numbered consecutively. Many of them by simple transformations, in some cases by cyclic changes, are transformed to later ones. To the remaining ones, all non-conflicting arrangements involving *126* are added in turn, and to these very numerous sets now containing four arrangements each, all arrangements of *127* are added in turn. But whenever any two arrangements in a set are capable of being transformed into a later one of the set of one hundred mentioned above, that set is discarded. Thus sets of four, five, and six arrangements are obtained, though by reason of conflicting triads their numbers do not increase as radidly as might be supposed. By taking note of the derivation of the individual arrangements involving *346*, *347*, etc., these transformations are often quickly discovered. The essential feature of the method is the transformation of the uncompleted sets to later sets of the "one hundred."

Some idea of the success with which this was accomplished is gained from the fact that the final solution—for there is apparently only one—was found three times, instead of once; while it might have been found as many times as there are ways of transforming the solution.

Problems 215 and 217 were also solved by L. E. Newcomb, Los Gatos, Cal. No. 117 was also solved by F. P. Matz.

219. Proposed by Dr. SAUL EPSTEEN, The University of Chicago.

Sum to infinity $\frac{1.2}{3}+\frac{2.3}{3^2}+\frac{3.4}{3^3}+\ldots\ldots$

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

The series is the expansion of $\frac{2}{3}(1-\frac{1}{3})^{-3}$, and the required sum is therefore $\frac{2}{3}(\frac{2}{3})^{-3}$ or $\frac{9}{4}$.

II. Solution by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

This is a recurring series whose general term is $n(n+1)x^n$. Since the scale of relation is $(1-x)^3$ in which $x=\frac{1}{3}$, we have $(1-x)^3S=2x$.

$\therefore S=2x/(1-x)^3=2\frac{1}{4}$.

Also solved by M. E. Graber, Grace M. Bareis, J. H. Meyer, J. Scheffer, L. E. Dickson, G. B. M. Zerr, and H. Heaton.

220. Proposed by L. ROBINSON, B. S., Philadelphia, Pa.

Find the sum of the first $n+1$ terms of the series

$$1+\frac{m}{1!}+\frac{m\,(m+1)}{2!}+\frac{m(m+1)(m+2)}{3!}+\ldots\ldots\ldots$$

Solution by HEATON HEATON, Atlantic, Iowa.

Writing $m+1$ for $m$, the series becomes

$$1+\frac{(m+1)}{1!}+\frac{(m+1)\,(m+2)}{2!}+\ldots\ldots+\frac{(m+1)(m+2)\ldots\ldots(m+n)}{n!}$$

$$=\frac{1}{1!}+\frac{2(m+1)}{2!}+\frac{3(m+1)\,(m+2)}{3!}+\ldots\ldots+\frac{n(m+1)(m+2)\ldots\ldots(m+n-1)}{n!}$$

$$+\frac{(m+1)(m+2)\ldots\ldots(m+n)}{n!}.$$

Transposing all the terms ot the second number but the last we obtain

$$1+\frac{m}{1!}+\frac{m\,(m+1)}{2!}+\ldots\ldots+\frac{m(m+1)\,(m+n-1)}{n!}=\frac{(m+1)(m+2)\ldots\ldots(m+n)}{n!}.$$

Also solved by Lloyd Holsinger, J. Scheffer, Grace M. Bareis, G. B. M. Zerr, L E. Dickson, G.W. Greenwood, F. P. Matz.

221. Proposed by F. P. MATZ, Ph. D., Sc. D.

Eliminate the unknowns from

$$x/y+y/z+z/x=a\ldots\ldots\ldots(1),\quad x/z+y/x+z/y=b\ldots\ldots\ldots(2),$$
$$(x/y+y/z)(y/z+z/x)(z/x+x/y)=c\ldots\ldots\ldots(3).$$

Solution by GRACE M. BAREIS, A. B., Bala, Pa.

Substituting in (3) the values from (1) it becomes

$$(a-z/x)(a-x/y)(a-y/z)=c,\text{ or }a^3-(z/x+x/y+y/z)^2+(z/y+y/x+x/z)a-1=c,$$

whence $ab-1=c$ or $ab-c-1=0$.

Also solved by J. Scheffer, Lloyd Holsinger, Henry Heaton, M. E. Graber, L. E. Dickson, G. B. M. Zerr, G W. Greenwood, and the Proposer.

222. Proposed by G. W. WALKER, Camden, N. J.

Extract the square root of $87-12\sqrt{42}$.

Solution by J. H. MEYER, S. J., Spring Hill College, Mobile, Ala.

Let $87-12\sqrt{42}=(\sqrt{\alpha}-\sqrt{\beta})^2$; then $\alpha+\beta=87$; $\alpha\beta=1512$; hence $\alpha=63$, $\beta=24$, and $\sqrt{\alpha}-\sqrt{\beta}=3\sqrt{7}-2\sqrt{6}$.

Also solved by Henry Heaton, Lloyd Holsinger, J. J. Keyes, M. E. Graber, L. E. Dickson, S. F. Norris, J. Scheffer, Grace M. Bareis, G. B. M. Zerr, G. W. Greenwood, L. S. Shively, and F. P. Matz.

## GEOMETRY.

247. Proposed by SETH PRATT, C. E., Tecumseh, Neb.

From two given points without a circle to draw two lines meeting in the circumference and making equal angles with the tangent at that point.

Solution by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Let $C$ be the center of the circle, $T_1PT_2$ the tangent to the circle at $P$; $P_1$ and $P_2$ the given points; $PC=r$, $\angle P_1PT_1=\angle P_2PT_2=\phi$, and the polar coördinates of $P_1$ and $P_2$, $(\rho_1, \theta_1)$ and $(\rho_2, \theta_2)$, respectively.

Then $P_1P_2=\sqrt{[\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\theta_1-\theta_2)]}$ .....(1).

Also, since $\angle P_1PC=\angle P_2PC=90°+\phi$,

$$P_1P=\sqrt{(\rho_1^2+r^2+2\rho r\sin\phi)} \text{ .....(2), and } P_2P=\sqrt{(\rho_2^2+r^2+2\rho_2 r\sin\phi)} \text{ .....(3).}$$

From the $\triangle P_1PP_2$, $\angle P_1PP_2=180°-2\phi$, we have

$$\cos(180°-2\phi)=-\cos 2\phi=\frac{(P_1P)^2+(P_2P)^2-(P_1P_2)^2}{2(P_1P)(P_2P)} \text{ .....(4).}$$

From (4) by means of (1), (2), (3), we have after putting $(\rho_1+\rho_2)=a$, $\rho_1\rho_2=b$, $(\rho_1^2+r^2)=c$, $(\rho_2^2+r^2)=d$, $\rho_1 r=e$, $\rho_2 r=f$, the following equation:

$$16ef\sin^6\phi+8(cf+de)\sin^5\phi+4(cd-4ef)\sin^4\phi-8(cf+de)\sin^3\phi$$
$$-(a^2r^2+4cd+4ef)\sin^2\phi+2[cf+de-ar^3-abr\cos(\theta_1-\theta_2)]\sin\phi$$
$$+[cd-r^4-b^2\cos^2(\theta_1-\theta_2)-2br^2\cos(\theta_1-\theta_2)]=0 \text{ .....(5),}$$

for the determination of $\sin\phi$.

Knowing $\sin\phi$, the value of $P_1P$ and $P_2P$ can be determined from (2) and (3).

Also solved by J. Scheffer.

248. Proposed by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, Ohio.

Given $AB$, $BC$ in a straight line, to produce it to $D$ so that $AD.CD=BD^2$.

I. Solution by GRACE M. BAREIS, A. B., Bala, Pa.

Describe a circle on $AC$ as diameter, and construct the diameter $EF$ perpendicular to $AC$. At the point $G$ where $EB$ (or $FB$) cuts the circle again, draw the tangent. This tangent will cut $AC$ at the required point, $D$.

For $\angle EGD$ is measured by $\frac{1}{2}(\text{arc}GD+\text{arc}DE)$ and $\angle GBD$ is measured by $\frac{1}{2}(\text{arc}GD+\text{arc}AE)$, since $\text{arc}DE=\text{arc}AE$, $\therefore \triangle GBD$ is isosceles, *i. e.*, $BD=GD$.

But $AD.CD=GD^2$. $\therefore AD.CD=BD^2$.

II. Solution by R. D. CARMICHAEL, Hartselle, Ala., and SAM I. JONES, Gunter, Texas.

Produce $AC$ indefinitely to $H$. Lay off $BM=AB$. Find the third pro-

portional to $CM$ and $BC$. From $C$ measure the distance $CD$=this third proportional. $D$ is the required point.

For $BC^2=CM.CD$, or $BC^2=(AB-BC)CD$. $\therefore CD.AB=BC^2+BC.CD$.

$$CD.AB+BC.CD+CD^2=BC^2+2BC.CD+CD^2.$$

$$CD(AB+BC+CD)=(BC+CD)^2.$$

$$AD.CD=BD^2.$$

Also solved by S. A. Corey, J. R. Hitt, F. D. Posey, M. E. Graber, W. W. Landis, and G. W. Greenwood.

249. Proposed by W. W. BEMAN, The University of Michigan.

Given the distances of a point in the plane of a square from three of its vertices, to find the side of the square.

250. Proposed by W. W. BEMAN, The University of Michigan.

Given the distances of a point in the plane of an equilateral triangle from the vertices; to find the side of triangle. [Perkins' *Geometry*, Olney's *Geometry*.]

I. Solution by F. D. POSEY, San Mateo, Calif.

Consider the general case, viz: Given the distances of a point in the plane of a regular $n$-gon to three consecutive vertices, to find the side of the $n$-gon.

Let the vertices be $A$, $B$, $C$ in order, say clockwise, and $P$ the given point. Let $PA$, $PB$, $PC\equiv a$, $b$, $c$, respectively. Let $\angle ABP=\alpha$, $\angle PBC=\beta$, taking these clockwise if $P$ be without the angle $ABC$, and counter-clockwise if $P$ be within. Call the side of the $n$-gon, $x$.

There are now two cases: (1) $P$ within the angle $ABC$ of the $n$-gon, (2) $P$ without this angle. In the first case $\alpha+\beta=\frac{n-2}{n}\pi$. $\therefore \cos\beta=-\cos\frac{2\pi}{n}\cos\alpha+\sin\frac{2\pi}{n}\sin\alpha$. $\therefore \sin\alpha=\operatorname{cosec}\frac{2\pi}{n}\cos\beta+\cot\frac{2\pi}{n}\cos\alpha$.

In the second case $\alpha+\beta=2\pi-\frac{n-2}{n}\pi$. $\therefore \sin\alpha=-\operatorname{cosec}\frac{2\pi}{n}\cos\beta-\cot\frac{2\pi}{n}\cos\alpha$

Now $\cos\alpha=\frac{b^2+x^2-a^2}{2bx}$ (when $c$ is between $b$ and $a$ we have $\cos(2\pi-\alpha)=\cos\alpha$), and $\cos\beta=\frac{b^2+x^2-c^2}{2bx}$ (when $a$ is between $b$ and $c$ we have $\cos(2\pi-\beta)=\cos\beta$). In both cases (1) and (2), $\sin^2\alpha+\cos^2\alpha=(\operatorname{cosec}\frac{2\pi}{n}\cos\beta+\cot\frac{2\pi}{n}\cos\alpha)^2+\cos^2\alpha=1$, which equation after substituting the above values for $\cos\alpha$ and $\cos\beta$ reduces to:

$$\left[\left(\cot\frac{2\pi}{n}+\operatorname{cosec}\frac{2\pi}{n}\right)^2+1\right]x^4+2\left\{\left(\cot\frac{2\pi}{n}+\operatorname{cosec}\frac{2\pi}{n}\right)\left[\cot\frac{2\pi}{n}(b^2-a^2)+\operatorname{cosec}\frac{2\pi}{n}(b^2-c^2)\right]-a^2-b^2\right\}x^2+\left[\cot\frac{2\pi}{n}(b^2-a^2)+\operatorname{cosec}\frac{2\pi}{n}(b^2-c^2)\right]^2+(b^2-a^2)^2=0.$$

In the case of the square, $n=4$. $\therefore \cot\frac{2\pi}{n}=0$, and $\operatorname{cosec}\frac{2\pi}{n}=1$.

The equation then becomes $2x^4-2(a^2+c^2)x^2+(b^2-a^2)^2+(b^2-c^2)^2=0$.

In the case of the equilateral triangle, $n=3$.

$\therefore \cot\frac{2\pi}{n}=-\frac{1}{\sqrt{3}}$, $\operatorname{cosec}\frac{2\pi}{n}=\frac{2}{\sqrt{3}}$, and we have $x^4-(a^2+b^2+c^2)x^2+a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2=0$.

These equations may be solved as quadratics and by taking a unit length the expression for $x$ may be constructed.

II. Solution by G. I. HOPKINS, Manchester, N. H.

Let $AB$, $CD$, and $EF$ be the distances from the three vertices, and $H$ the given vertical angle. Draw $\angle JKL=\angle H$ and make its sides both equal to $AB$, or any one of the three lengths. Then with $J$ and $L$ as centers, and $CD$ and $EF$ as radii fix the point $N$. Join $NK$. Then construct the isosceles $\triangle NKO$, $NK$ being one of the equal sides, and vertical $\angle NKO=\angle H$.

$\therefore NOK$ is the required triangle. For, join $JO$. $JK=LK$, $NK=OK$, $\angle NKL=\angle JKO$. $\therefore \triangle JOK=\triangle NLK$. $\therefore NL=OJ$. $\therefore \triangle NKO$ is isosceles with vertical $\angle H$, and the three distances $JK$, $JO$, and $JN$ are equal, respectively, to $AB$, $CD$, and $EF$.

If the triangle is equilateral, then $\angle H$ is two-thirds of a right angle, and we have the same construction.

Since a diagonal of a square divides it into two isosceles triangles whose vertical angles are 90°, therefore if $\angle H=90°$ we have the same construction and $NK$ is a side of the required square.

Also solved by Henry Heaton, W. W. Landis, J. R. Hitt, M. E. Graber, and G. W. Greenwood.

## CALCULUS.

189. Proposed by J. E. SANDERS, Hackney, Ohio.

Solve $d^2y/dx^2=-\beta^2(p+y)$, $p$ and $\beta$ being constants. The initial conditions are $y=0$ for $x=0$, $l$; $dy/dx=0$ for $x=l/2$. [Merriman's *Mechanics*, 9th Ed., 1903, §62.]

I. Solution by W. D. STAYTON, Student, Louisiana State University, Baton Rouge, La.; HOWARD L. STOKER, Lehigh University, and T. O. WHITAKER.

Multiplying by $2dy$ and integrating the resulting equation, we have

$$\left(\frac{dy}{dx}\right)^2=-2\beta^2y-\beta^2y^2+c_1 \quad \ldots\ldots (1).$$

Since $dy/dx=0$, for $y=\triangle$, $c_1=\beta^2\triangle^2+2\beta^2p\triangle$. Substituting this value of $c_1$ and extracting the square root, we have

$$dy/dx=\beta\sqrt{(-2py-y^2+\triangle^2+2\triangle p)}=\beta\sqrt{[(\triangle+p)^2-(y+p)^2]}.$$

$$\therefore \beta dx = \frac{dy}{\sqrt{[(\triangle+p)^2-(y+p)^2]}} = \frac{\frac{dy}{\triangle+p}}{\sqrt{1-\left(\frac{y+p}{\triangle+p}\right)^2}}.$$

$$\therefore \beta x = \sin^{-1}\left(\frac{y+p}{\triangle+p}\right)+c_2 \quad \text{.....(2).}$$

Since $y=0$, for $x=l$, $c_2=\beta l-\sin^{-1}\left(\frac{p}{\triangle+p}\right)$.

$$\therefore \beta x = \sin^{-1}\left(\frac{y+p}{\triangle+p}\right)+\beta l-\sin^{-1}\left(\frac{p}{\triangle+p}\right) \quad \text{.....(3).}$$

Since $y=\triangle$ for $x=\frac{1}{2}l$, we have by substituting in (3) and solving for $\triangle$,

$$\triangle = p(\sec\frac{\beta l}{2}-1).$$

Substituting this value of $\triangle$ in (3), and reducing, we have

$$y+p=p\left(\frac{\sin\beta(l-x)+\sin\beta x}{\sin\beta l}\right).$$

II. Solution by SAUL EPSTEEN, Ph. D., The University of Chicago, and S. A. COREY, Hiteman, Iowa.

Substituting $z=p+y$, the equation takes the form $d^2z/dx^2=-\beta^2 z$, of which $z=c_1\sin\beta x+c_2\cos\beta x$ ($c_1$ and $c_2$ being arbitrary constants) is the most general solution. The initial conditions are now $z=p$ for $x=0, l$; $dz/dx=0$, for $x=\frac{1}{2}l$. From the first, we have $c_2=p$, $c_1=p\tan\beta\frac{1}{2}l$, and therefore

$$y=-p+p\tan\beta\tfrac{1}{2}l\sin\beta x+p\cos\beta x.$$

It can now be verified that the condition $z=0$, for $x=l$ is fulfilled and is therefore superfluous in the enunciation of the problem.

The solution may evidently be transformed to the following form:

$$y=\frac{p(\cos\beta\frac{1}{2}l+\sin\beta\frac{1}{2}l\,\sin\beta x+\cos\beta\frac{1}{2}l\,\cos\beta x)}{\cos\beta\frac{1}{2}l}=\frac{p}{\cos\beta\frac{1}{2}l}[-\cos\beta\tfrac{1}{2}l+\cos\beta(x-\tfrac{1}{2}l)].$$

Also solved by G. W. Greenwood, G. B. M. Zerr, J. O. Mahoney, and W. W. Landis.

190. Proposed by SAUL EPSTEEN, The University of Chicago, Chicago, Ill.

$$\int_0^\infty \frac{\sin x\cos\beta x}{x}dx, \quad \int_0^\infty \frac{\sin\alpha x\cos x}{x}.$$

Solution by IDA M. SCHOTTENFELS, M. A., New York, N. Y.

$\int_0^\infty \frac{\sin x \cos\beta x}{x} dx = \frac{1}{2}\pi, \frac{1}{4}\pi, 0$, according as $-1<\beta<+1, \beta=\pm 1, +1<\beta<-1$.

By making use of the trigonometrical formula

(9) $$\sin x \cos\beta x = \tfrac{1}{2}[\sin(1+\beta)x + \sin(1-\beta)x]$$

$$\int_0^\infty \frac{\sin x \cos\beta x}{x} dx = A + B = \tfrac{1}{2}\left[\int_0^\infty \frac{\sin(1+\beta)x}{x} dx + \int_0^\infty \frac{\sin(1-\beta)x}{x} dx\right].$$

$A = \int_0^\infty \frac{\sin(1+\beta)x}{x} dx = +\frac{1}{2}\pi,\ 0,\ -\frac{1}{2}\pi$ according as $(1+\beta)>0, (1+\beta)=0, (1+\beta)<0$.

$B = \int_0^\infty \frac{\sin(1-\beta)x}{x} dx = +\frac{1}{2}\pi,\ 0,\ -\frac{1}{2}\pi$ according as $(1-\beta)>0, (1-\beta)=0, (1-\beta)<0$.

$A+B=\frac{1}{2}\pi$ where $-1<\beta<+1$. $A+B=\frac{1}{4}\pi$ where $\beta=\pm 1$. $A+B=0$ where $+1<\beta<-1$.

$\int_0^\infty \frac{\sin\alpha x \cos x}{x} dx = \frac{1}{2}\pi,\ \frac{1}{4}\pi,\ 0,\ -\frac{1}{4}\pi,\ -\frac{1}{2}\pi$ according as $\alpha>1, \alpha=1, -1<\alpha<+1, \alpha=-1, \alpha<-1$.

By making use of the trigonometrical formula

(8) $$\sin\alpha x \cos x = \tfrac{1}{2}[\sin(\alpha+1)x + \sin(\alpha-1)x]$$

$$\int_0^\infty \frac{\sin\alpha x \cos x}{x} dx = C + D = \tfrac{1}{2}\left[\int_0^\infty \frac{\sin(\alpha+1)x}{x} dx + \int_0^\infty \frac{\sin(\alpha-1)x}{x} dx\right].$$

$C = \int_0^\infty \frac{\sin(\alpha+1)x}{x} dx = \frac{1}{2}\pi,\ 0,\ -\frac{1}{2}\pi$ according as $(\alpha+1)>0, (\alpha+1)=0, (\alpha+1)<0$.

$D = \int_0^\infty \frac{\sin(\alpha-1)x}{x} dx = \frac{1}{2}\pi,\ 0,\ -\frac{1}{2}\pi$ according as $(\alpha-1)>0, (\alpha-1)=0, (\alpha-1)<0$.

$C+D=\frac{1}{2}\pi$ where $\alpha>1$. $C+D=\frac{1}{4}\pi$ where $\alpha=1$. $C+D=0$ where $-1>\alpha<+1$. $C+D=-\frac{1}{4}\pi$ where $\alpha=-1$. $C+D=-\frac{1}{2}\pi$ where $\alpha<-1$.

We now set the integral in (I) and (II) equal to a function of the parameters $\alpha$ and $\beta$ thus

$$\int_0^\infty \frac{\sin\alpha x \cos x}{x} dx = \psi(\alpha), \quad \int_0^\infty \frac{\sin x \cos\beta x}{x} dx = \chi(\beta),$$

and trace the resulting curves.

The finite discontinuities occur in both cases for the parameters $\alpha, \beta = \pm 1$.

Also solved by M. E. Graber, G. W. Greenwood, F. D. Posey, S. A. Corey, G. B. M. Zerr, and J. O. Mahoney.

## MECHANICS.

170. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Prove that the moment of inertia of an ogival head rotating about its geometrical axis is $\frac{\pi w}{g}\int_0^{R\sqrt{(4n-1)}} y^4 dx$, where $w$ is the weight in pounds of a cubic foot of material, $R$ the radius of the base of the ogive, and $n$ the diameter of projectile.

Solution by LYTLE BROWN, Lehigh University, and the PROPOSER.

An ogival head is one-half the solid generated by the revolution of a segment of a circle about its chord. The equation of the generating curve is $y = \sqrt{(4n^2R^2 - x^2)} - (2n-1)R$, the origin being the center of the base of the ogive. If $y=0$, the resulting value of $x$ is the length of the head, that is, the length of the head is $R\sqrt{(4n-1)}$. The moment of inertia of an elementary cylinder about the axis of $x$ is $4\pi y^2 \frac{\omega}{g} . y^2 dx$, or $dI = \frac{\pi\omega}{2g} y^4 dx$.

$$\therefore I = \frac{\pi\omega}{g}\int_0^{R(\sqrt{4n-1})} y^4 dx.$$

## AVERAGE AND PROBABILITY.

155. Proposed by E. B. WILSON, Ph. D., Yale University.

The game of craps is played with two dice. If the player throws 7 or 11 on the first throw he wins. If he throws 12, 2, or 3 he loses. If the player throws any other number, that is to say, 4, 5, 6, 8, 9, 10, he is obliged to continue throwing until he throws that number again or until he throws 7. If he succeeds in throwing his first throw before he does 7, he wins—otherwise he loses. Required the odds against him. (Note that he can continue throwing indefinitely without getting either his original throw or the 7).

REMARK. A correct solution of this problem is given by Dr. Zerr, in Vol. X, No. 3, p. 81. Mr. J.E. Sanders sent in a different solution, his answer being $\frac{244}{251}$, in favor of the player.

Problem 131 should be numbered 158. Three solutions of this problem have been received, none of which agree.

Problem 157 should be numbered 159.

## MISCELLANEOUS.

144. Proposed by IRA M. DeLONG, Professor of Mathematics in the University of Colorado, Boulder, Col.

Determine the number of distinct spherical polygons of $n$ sides formed by arcs of $n$ given great circles on a sphere, each arc to be greater than 0 degrees, and less than 360 degrees, and no two sides of any polygon to lie an the same circle.

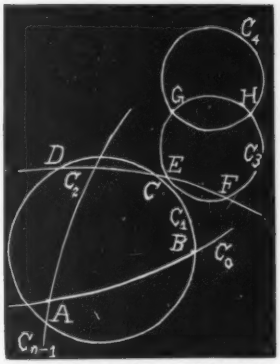
Solution by ARNOLD EMCH, Professor of Graphics and Mathematics, University of Colorado.

1. No two sides of any of the required polygons must lie on the same circle. Arcs between 0 and 360° are admitted.

2. Designate the $n$ great circles by $c_0, c_1, c_2, c_3, \ldots\ldots, c_{n-1}$. Take any of the circles, say $c_0$, and on it all arcs which may be taken as sides of required polygons. Construct all possible polygons of $n$ sides, $x$ in number, each having one of these arcs as a side.

Since each of all possible polygons of $n$ sides on the sphere has one of its sides on $c_0$, it is clear that the number of these polygons is identical with $x$.

There are $2(n-1)$ points of intersection of $c_0$ with the $(n-1)$ remaining circles. These may be combined in $\{2(n-1)[2(n-1)-1]\}/2$ ways to form arcs of the required polygons. But as no two arcs shall be on one circle we have to exclude from this number those arcs which are formed on $c_0$ by each of the $(n-1)$ circles separately; *i. e.*, $(n-1)$. The number of arcs on $c_0$, each of which may be taken as a side of a polygon, is therefore $(n-1)[2(n-1)-1]-(n-1)$, or, since we can also take their supplements to 360°,




$$2(n-1)[2(n-1)-1]-2(n-1)=2^2(n-1)(n-2).$$

4. Take now one of these arcs $AB$ and the two circles $c_1$ and $c_{n-1}$ through $B$ and $A$. The question is, in how many different ways can I pass from $A$ to $B$, from $B$ on $c_1$ to points of arcs of $c_2$, $c_3$, ....., $c_{n-2}$; then on arcs of these $c$'s to points on $c_{n-1}$ and back to $A$. For this purpose take any of the permutations of the $c$'s, say $c_2, c_3, c_4, \ldots\ldots, c_{n-2}$, and let $c_1$ cut $c_2$ in $C$ and $D$, $c_2$ cut $c_3$ in $E$ and $F$, $c_3$ cut $c_4$ in $G$ and $H$, ....., $c_{n-2}$ cut $c_{n-1}$ in $X$ and $Y$, as indicated in the adjoining diagram.

From $B$, I can reach $C$ and $D$ each in two ways; from $C$ and $D$, I can, on $c_2$, reach each of $E$ and $F$ in two ways; so that from $B$, $E$, and $F$ may each be reached in $2^3$ ways, or both in $2^4$ ways. This process continued may be illustrated by the following table.

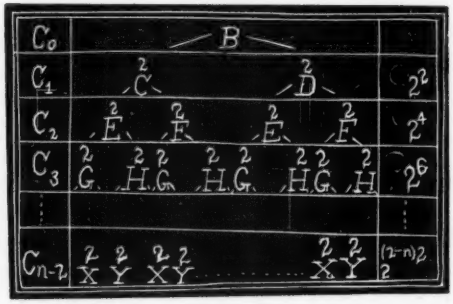
In this table the first column contains the circles on which the points indicated in the same row are reached. For instance, the points $G$ and $H$ are reached from $E$ and $F$ by describing arcs on the circle $c_3$. The last column shows in how many ways the points in each row as a whole may be reached from $B$. Thus all $G$'s and $H$'s in the fourth row may be reached from $B$ in $2^6$ different ways. From the table it is seen that the points $X$ and $Y$ which are also on the circle $c_{n-1}$ may be reached from $B$ in $2^{2n-4}$ different ways. From $X$ and $Y$ each, $A$ may be reached in two ways; hence $A$ may be reached in the prescribed manner in $2^{2n-3}$ different ways. The same number is obtained for every permutation of $c_2, c_3, \ldots\ldots, c_{n-2}$; so that there are $(n-3)!2^{2n-3}$ polygons having $AB$ as a side.

5. As there are $2^2(n-1)(n-2)$ different arcs on $c_0$, the total number $x$ of polygons is

$$2^2(n-1)(n-2)(n-3)\,!\,2^{2n-3},$$

$$\text{or } x=(n-1)\,!\,2^{2n-1}.$$

6. To find the total number of all possible spherical polygons that may be formed by arcs of $n$ great circles, it is clear that there will be $(r-1)\,!\,2^{2r-1}$ polygons of $r$ sides, and consequently




$$\sum_{r=3}^{r=n}[{}^nC_r.(r-1)\,!\,2^{2r-1}]$$

will be the number of all possible polygons.

7. *Examples.* With three great circles on a sphere $(3-1)\,!\,2^{3-1}=64$ triangles may be formed. The number of spherical triangles and quadrangles which may be formed by four great circles is

$$(4-1)\,!\,2^{3-1}+{}^4C_3.(3-1)\,!\,2^{3-1}=768+4.64=1024.$$

## PROBLEMS FOR SOLUTION.

### ALGEBRA.

226. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Find the real roots of the system

$$x^2+w^2+v^2=a^2,\quad vw+u(y+z)=bc,$$
$$w^2+y^2+u^2=b^2,\quad wu+v(z+x)=ca,$$
$$v^2+u^2+z^2=c^2,\quad uv+w(x+y)=ab.$$

227. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Solve $x+y+xy+x^2y+xy^2+x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3=11$; $x^4y+3x^3y^2+3x^2y^3+2x^4y^2+4x^2y^3+2x^2y^4+4x^4y^3+4x^3y^4+xy^4+x^5y^2+x^5y^3+2x^4y^4+x^2y^5+x^3y^5=30$.

### GEOMETRY.

251. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Represent the vertices of any regular polygon by the consecutive numbers 1, 2.....$p$.....$q$.....$r$.....$n$. To find the sides and area of the triangle formed by joining $p$, $q$, and $r$.

252. Proposed by FREDERICK R. HONEY, Ph. B., Trinity College, Hartford, Conn.

Two plane mirrors form an angle which is less than 45°. Any two points are assumed within this angle in a plane perpendicular to the intersection of the mirrors. A ray of light passes through one point, and after being reflected twice at each mirror, it passes through the second point. Find the path of the ray.

253. Proposed by SAM I. JONES, Gunter Bible College, Gunter, Texas.

The number of cubic inches contained by two equal opposite spherical segments, together with the number of cubic inches contained by the cylinder included between these segments, is 600; if this be $\frac{2}{3}$ of the number of cubic inches contained by the whole sphere, find the height of the cylinder.

## CALCULUS.

191. Proposed by J. E. SANDERS, Hackney, Ohio.

A fly goes along a radius of a moving carriage wheel from center to circumference while the wheel makes $n$ revolutions. If each move uniformly, what is the equation to the curve described by the fly in space, and what is its length when the wheel has made $1/m$ of a revolution?

192. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Show that the volume $V$ of the hyper-ellipsoid with semi-axes $a_1$, $a_2$, $a_3$, $a_4$, etc., in space of $2n$ and $2n+1$ dimensions is

$$V_{2n}=\frac{a_1.a_2.a_3.....a_{2n}.\pi^n}{1.2.3.4.5.....n}; \qquad V_{2n+1}=\frac{2^{n+1}.a_1.a_2.a_3.....a_{2n+1}.\pi^n}{1.3.5.7.9.....(2n+1)}.$$

193. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the eccentricity of the maximum semi-ellipse inscribed in a given isosceles triangle.

## MECHANICS.

173. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Oklahoma.

A squirrel is in a cylindrical cage and oscillating with it about its axis which is horizontal. At the instant when he is at the highest point of the oscillation, he leaps to the opposite extremity of the diameter and arrives there at the same instant as the point at which he left. Determine his leap completely.

174. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

By what per cent. is the striking force of a hailstone increased in falling 1000 feet through a stratum of atmosphere moving uniformly eastward at the rate of 60 miles an hour?

## DIOPHANTINE ANALYSIS.

124. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find (1) three square numbers whose sum is a cube; (2) three cube numbers whose sum is a square.

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

What *values* of $x$ will make $E=\frac{(x+7)(x+5)}{(x-7)(x-5)}$ represent square numbers?

## AVERAGE AND PROBABILITY.

160. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Oklahoma.

Two points are taken at random in a triangle, the line joining them dividing the triangle into two portions. Find the mean value of that portion containing the center of gravity.

161. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

A triangle is inscribed at random in a circle; ($a$) what is the chance the triangle is *oblique;* and ($b$) what is the chance the triangle is *less in area* than $\frac{1}{4}\pi r^2$?

## MISCELLANEOUS.

146. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

The year 1905 *began*, and will *end*, on a Sunday. Prove that this can not occur again until the year 2015.

147. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If an *unknown* curve be described under a constant acceleration not tending to the center and the hodograph is a cardioid, what is the unknown curve?

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

## *BOOKS.*

*Academic Algebra.* By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in Teachers College, Columbia University, New York. 8vo. Half Leather, 383 pages. Price, $1.25. Boston and Chicago: Ginn & Co.

A splendid work for Academies and High Schools. B. F. F

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# APPLICATION OF SEVERAL THEOREMS IN NUMBER THEORY TO GROUP THEORY.

By DR. G. A. MILLER, Stanford University.

Two previous notes (MONTHLY, Vol. 11, p. 129, and Vol. 12, p. 41) were devoted to exhibiting the advantages of employing elementary theorems in group theory in the proof of some of the fundamental theorems of number theory. The present note, on the contrary, aims to translate into the language of group theory several other important theorems of number theory. The object is to make these theorems more available to the student of group theory rather than to emphasize the advantages resulting from the employment of group theory concepts in the study of number theory. As most of the developments relate to ground which is decidedly common to the two subjects it is hoped that some of the results are also useful from the standpoint of number theory.

It is well known that a cyclic substitution of even order generates a group ($C$) in which half the substitutions are even and the other half are odd. The even substitutions are squares and the odd substitutions are non-squares under $C$.* As any substitution ($s$) has the same square as $s$ multiplied by the substitution of order 2 contained in $C$, it follows that the two square roots of any even substitution are either both odd or both even when the order of $C$ is divisible by 4. When this order is not divisible by 4 one of these square roots is even and the other is odd.

Let $C$ be the group of isomorphisms of the cyclic group of order $p$, $p$ being any odd prime. With each substitution of $C$ may be associated a number which is congruent (mod $p$) to the index of the power into which this substitution

*The even substitutions constitute a subgroup whose order is half the order of $C$.